Simulating Optimal Multi-Dimensional Auctions

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Introduction

Belloni et al. (2010) challenge the view that in the multi-dimensional setting optimal auctions are complex. We extend their simulation results and evaluate three conjectures derived from their claims.

Model

- 1 good with $j = 1, \dots, J$ quality grades and $i = 1, \dots, N$ buyers
- Each buyer i has a multi-dimensional type given by $v^i = (v^i_1, \dots, v^i_J)$. The buyer's types are distributed according to F with support $V \equiv \prod_{j \in J}[\underline{v}_j, \overline{v}_j]$. The valuations are independent across buyers but could be correlated across grades.
- Buyer i's utility is given by $u^i = \sum_{j=1}^J v^i_j q^i_j - m^i$
- A mechanism (q, m) is **incentive compatible** if

$$U^i(v^i, v^{-i}) \geq U^i(\widehat{v}^i, v^{-i}) \quad \text{for all } \widehat{v}, v \in V \text{ and } i \in N. \qquad \text{(IC)}$$

- A mechanism (q, m) is **individually rational** if

$$U^i(v^i, v^{-i}) \geq 0 \quad \text{for all } v \in V \text{ and } i \in N. \qquad \text{(IR)}$$

- The optimization problem for the seller can be formulated:

$$\begin{cases} \max_{Q,U} \int_V \sum_{j\in J}(v_j - c_j)Q_j(v) - U(v)dF(v) \\ Q_j(v) \geq 0 \quad \text{for all } v \in V, j \in J \\ N\int_A \sum_{j \in J} Q_j(v)dF(v) \leq 1 - \left(\int_{V\setminus A} dF(v)\right)^N \quad \text{for all } A \subset V \\ U(v) - U(\widehat{v}) \geq \sum_j Q_j(\widehat{v})(v_j - \widehat{v}_j) \quad \text{for all } \widehat{v}, v \in V \\ U(\underline{v}) = 0 \end{cases} \qquad (P^*)$$

where $U(v) \equiv U(v, v)$ and $Q^i_j(v^i) = \int_{V^{N-1}} q^i_j(v^i, v^{-i})dF^{-i}(v^{-i})$.

Conjectures & Refutations

Belloni et al. (2010) propose the class of **exclusive buyer mechanisms** that perform well relative to the numerical optimal mechanisms in their simulations. In these mechanisms, buyers compete in a second price auction for the right to be the only buyer to get the object. The winner then buys object quality q_j at price p_j. In the **exclusive buyer mechanisms** buyer i with highest value $\beta^i = \max(v^i_1 - p_1, \dots, v^i_J - p_j)$ wins. Our conjecture is:

Conjecture 1. *The optimal auction revenue can be well approximated by the optimal exclusive buyer mechanism.*

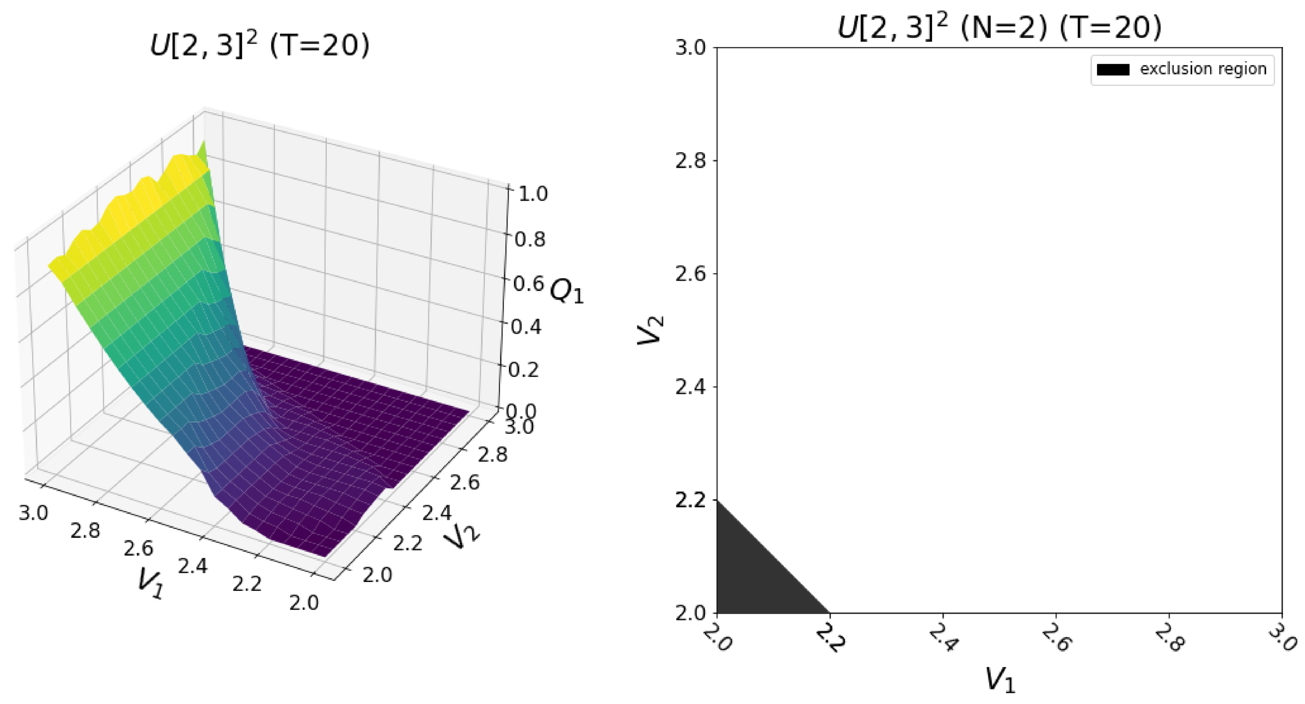

FIGURE 1: We test conjecture 1 in the setting where $N = 2$ buyers' types are uniformly distributed on $J = 2$ qualities $[2,3] \times [2,3]$ and costs are 0. The first figure shows the interim allocation probabilities for quality 1 (Q_1). The second figure depicts the exclusion region – the set of buyer values that receive object with zero probability.

The allocation rule in Figure 1 cannot be implemented by an **exclusive buyer mechanism** and so we **reject conjecture 1**.

Conjecture 2. *In some multi-buyer settings, the measure of buyers' types that receive no object (the exclusion region) in the optimal auction has measure zero in contrast to the corresponding single-buyer setting.*

The graphs in Figure 2 prompt us to **reject conjecture 2**.

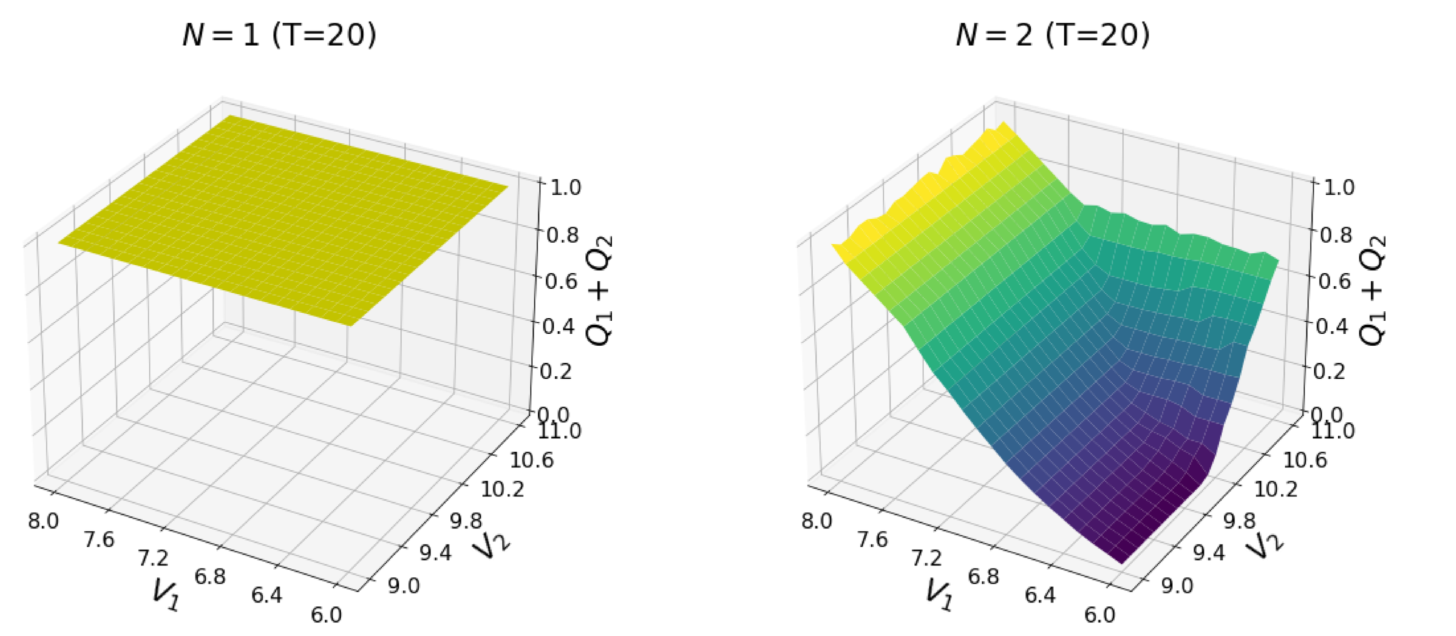

FIGURE 2: We test conjecture 2 in the original setting of Belloni et al. (2010). $N = 2$ buyers' types are uniformly distributed on $J = 2$ qualities $[6,8] \times [9,11]$ and costs are $c_1 = 0.9, c_2 = 5$. The first graph shows the allocation for quality 1 (Q_1) and quality 2 (Q_2) for the single-buyer case. The second figure shows the allocation for quality 1 (Q_1) and quality 2 (Q_2) for the two-buyer case.

Conjecture 3. *The exclusion region of the optimal allocation is independent of the number of buyers $N = 1, 2, 3, \dots$*

This conjecture was first proposed buyers by Kushnir and Shourideh Kushnir and Shourideh (2022) in the setting of Armstrong (1996). In Figure 3 the exclusion region did not change in any of the cases considered above for $N = 1, 2, 3$ buyers.

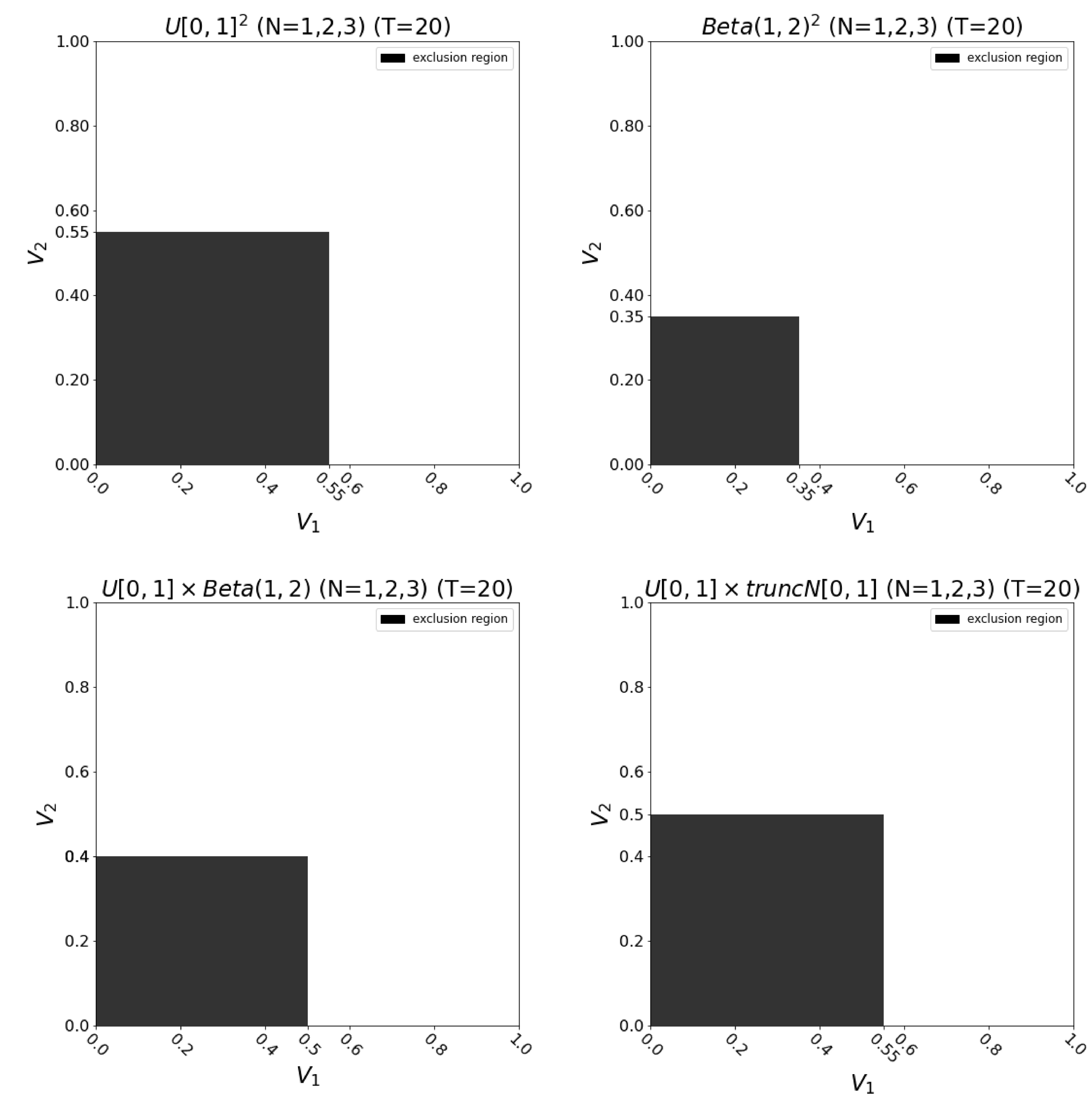

FIGURE 3: The top row depicts **symmetric** auctions: the left graph shows the exclusion region for $N = 1, 2, 3$ buyers with uniformly distributed types on $[0,1]$ and the right graph is instead for buyers with types distributed $Beta(1,2)$ on $[0,1]$. The bottom row shows **non-symmetric** auctions (still on $[0,1]$): the left graph $F(v_1) \sim U[0,1], F(v_2) \sim Beta(1,2)$ and for the right graph $F(v_1) \sim U[0,1]$ whereas $F(v_2)$ is a truncated normal distribution.

Therefore we consider conjecture 3 a promising area of future research and **do not reject conjecture 3**.

Conclusion

- We showed that the class of **exclusive buyer mechanisms** proposed by Belloni et al. (2010) does not approximate the optimal revenue in some environments.
- Additionally, the absence of an exclusion region in Belloni et al.'s result is not due to differences between optimal auctions in the single-buyer and multiple-buyer case.
- Lastly, we observed that the exclusion region is independent of the number of buyers in this setting. This observation is novel and should guide further theoretical work in this area.

References

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